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MATHEMATICS

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CIRCLE

THEORY AND EXERCISE BOOKLET

CONTENTS

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JEE Syllabus :

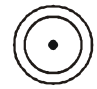
Equation of a circle in various forms, equations of tangent, normal and chord, parametric equations of a circle, intersection of a circle with a straight line or a circle, equation of a circle through the points of intersection of two circles and those of a circle and a straight line.

A. (1) DEFINITION

A circle is the locus of a point which moves in a plane in such a way that its distance from a fixed point remains constant. The fixed point is called the centre of the circle and the constant distance is called the radius of the circle.

(2) BASIC THEOREMS AND RESULTS OF CIRCLES

(a) Concentric circles : Circles having same centre.



(b) Congruent circles : Iff their radii are equal.

(c) Congruent arcs : Iff they have same degree measure at the centre.

Theorem 1 :

(i) If two arcs of a circle (or of congruent circles) are congruent, the corresponding chords are equal.
Converse : If two chords of a circle are equal then their corresponding arcs are congruent.

(ii) Equal chords of a circle (or of congruent circles) subtend equal angles at the centre.
Converse : If the angle subtended by two chords of a circle (or of congruent circle) at the centre are equal, the chords are equal.

Theorem 2 :

(i) The perpendicular from the centre of a circle to a chord bisects the chord.
Converse : The line joining the mid point of a chord to the centre of a circle is perpendicular to the chord.

(ii) Perpendicular bisectors of two chords of a circle intersect at tis centre.

Theorem 3 :

(i) There is one and only one circle passing through three non collinear points.

(ii) If two circles intersects in two points, then the line joining the centres is perpendicular bisector of common chords.

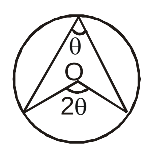
Theorem 4 :

(i) Equal chords of a circle (or of congruent circles) are equidistant from the centre.
Converse : Chords of a circle (or of congruent circles) which are equidistant from the centre are equal.

(ii) If two equal chords are drawn from a point on the circle, then the centre of circle will lie on angle bisector of these two chords.

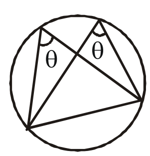
(iii) Of any two chords of a circle larger will be near to centre.

Theorem 5 :

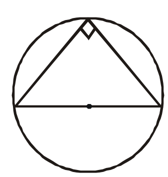


(i) The degree measure of an arc or angle subtended by an arc at the centre is double the angle subtended by it at any point of alternate segment.

(ii) Angle in the same segment of a circle are equal.



(iii) The angle in a semi circle is right angle.



Converse : The arc of a circle subtending a right angle in alternate segment is semi circle.

Theorem 6 :
An angle subtended by a minor arc in the alternate segment is acute and any angle subtended by a major arc in the alternate segment is obtuse.

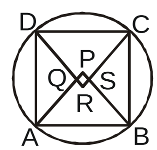
Theorem 7 :
If a line segment joining two points subtends equal angles at two other points lying on the same side of the line segment, the four points are concyclic, i.e. lie on the same circle.

(d) Cyclic Quadrilaterals : A quadrilateral is called a cyclic quadrilateral if its all vertices lie on a circle.

Theorem 1 :
The sum of either pair of opposite angles of a cyclic quadrilateral is 180º.
OR
The opposite angles of a cyclic quadrilateral are supplementary.
Converse : If the sum of any pair of opposite angle of a quadrilateral is 180º, then the quadrilateral is cyclic.

Theorem 2 :
If a side of a cyclic quadrilateral is produced, then the exterior angle is equal to the interior opposite angle.

Theorem 3 :
The quadrilateral formed by angle bisectors of a cyclic quadrilateral is also cyclic.



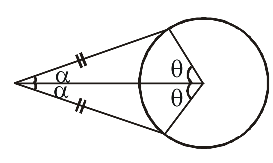
Theorem 4 :
If two sides of cyclic quadrilateral are parallel then the remaining two sides are equal and the diagonals are also equal.
OR
A cyclic trapezium is isosceles and its diagonals are equal.
Converse : If two non-parallel sides of a trapezium are equal then it is cyclic.
OR
An isosceles trapezium is always cyclic,

Theorem 5 :
The bisectors of the angles formed by producing the opposite sides of a cyclic quadrilateral (provided they are not parallel), intersect at right angle.

(3) TANGENTS TO A CIRCLE

Theorem 1 :
A tangent to a circle is perpendicular to the radius through the point of contact.
Converse : A line drawn through the end point of a radius and perpendicular to it is a tangent to the circle.

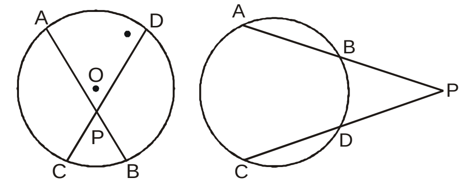
Theorem 2 :
If two tangents are drawn to a circle from an external point, then :
(i) they are equal
(ii) the subtend equal angles at the centre,
(iii) they are equally inclined to the segment, joining the centre to that point.



Theorem 3 :

If two chords of a circle intersect inside or outside the circle when produced, the rectangle formed by the segments of one chord is equal in area to the rectangle formed by the two segments of the other chord

$PA \times PB = PC \times PD$

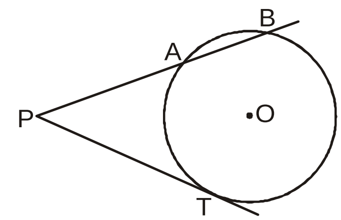


Theorem 4 :

If PAB is a secant to a circle intersecting the circle at A and B and PT is tangent segment, the $PA \times PB = PT^2$

OR

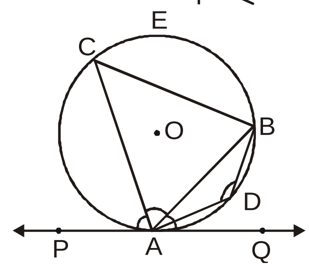
Area of the rectangle formed by the two segments of a chord is equal to the area of the square of side equal to the length of the tangent from the point on the circle.



Theorem 5 :

If a chord is drawn through the point of contact of tangent to a circle, then the angles which this chord makes with the given tangent are equal respectively to the angles formed in the corresponding alternate segments.

$\angle BAQ = \angle ACB$ and $\angle BAP = \angle ADB$



Converse :

If a line is drawn through an end point of a chord of a circle so that the angle formed with the chord is equal to the angle subtended by the chord in the alternate segment, then the line is a tangent to the circle.

B. STANDARD EQUATIONS OF THE CIRCLE

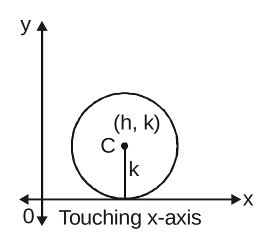
(a) Central Form :

If (h, k) is the centre and r is the radius of the circle then its equation is $(x - h)^2 + (y - k)^2 = r^2$

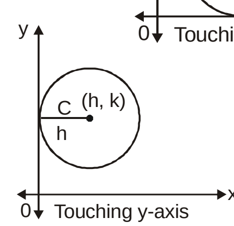
Special Cases :

(i) If centre is origin (0, 0) and radius is 'r' then equation of circle is $x^2 + y^2 = r^2$ and this is called the standard form.

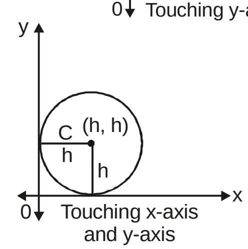
(ii) If radius of circle is zero then equation of circle is $(x - h)^2 + (y - k)^2 = 0$.
Such circle is called zero circle or point circle.



(iii) When circle touches x-axis then equation of the circle is
$(x - h)^2 + (y - k)^2 = k^2$.
or $x^2 + y^2 - 2hx - 2ky + h^2 = 0$



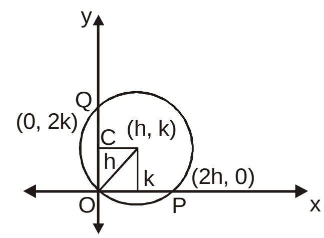
(iv) When circle touches y-axis then equation of the circle is
$(x - h)^2 + (y - k)^2 = h^2$
or $x^2 + y^2 - 2hx - 2ky + k^2 = 0$



(v) When circle touches both the axes (x-axis and y-axis) then equation of the circle is
$(x - h)^2 + (y - h)^2 = h^2$
or $x^2 + y^2 - 2hx - 2hy + h^2 = 0$

(vi) When circle passes through the origin and centre of the circle is (h, k) then radius $\sqrt{h^2+k^2}$ = r and intercept cut on x-axis OP = 2h, and intercept cut on y-axis is OQ = 2k and equation of circle is

$(x - h)^2 + (y - k)^2 = h^2 + k^2$ or $x^2 + y^2 - 2hx - 2hy = 0$



Note : Circle may exist in any quadrant hence for general cases use ± sign before h and k.

(b) General equation of circle :

$x^2 + y^2 + 2gx + 2fy + c = 0$. where g, f, c are constants and centre is (–g, –f)

i.e. $\left(-\frac{\text{coefficient of } x}{2}, -\frac{\text{coefficient of } y}{2}\right)$ and radius $r = \sqrt{g^2+f^2-c}$

Notes :

(i) If $(g^2 + f^2 - c) > 0$, then r is real and positive and the circle is a real circle.

(ii) If $(g^2 + f^2 - c) = 0$, then radius r = 0 and circle is a point circle.

(iii) If $(g^2 + f^2 - c) < 0$, then r is imaginary then circle is also an imaginary circle with real centre.

(iv) $x^2 + y^2 + 2gx + 2fy + c = 0$, has three constants and to get the equation of the circle at least three conditions should be known $\Rightarrow$ A unique circle passes through three non collinear points.

(v) The general quadratic equation in x and y, $ax^2 + by^2 + 2hxy + 2gx + 2fy + c = 0$ represents a circle if :

- coefficient of x^2 = coefficient of y^2 or $a = b \neq 0$
- coefficient of xy = 0 or h = 0
- $(g^2 + f^2 - c) \geq 0$ (for a real circle)

(c) Intercepts cut by the circle on axes :

The intercepts cut by the circle $x^2 + y^2 + 2gx + 2fy + c = 0$ on :

(i) x-axis = $\left|2\sqrt{g^2-c}\right|$ **(ii)** y-axis = $\left|2\sqrt{f^2-c}\right|$

Notes :

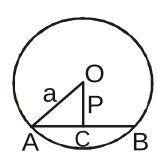
(i) If the circle cuts the x-axis at two distinct point then $g^2 - c > 0$

(ii) If circle touches x-axis then $g^2 = c$.

(iii) If circle touches y-axis $f^2 = c$.

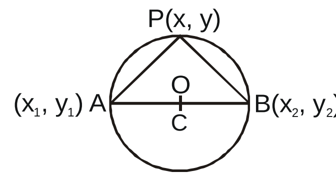
(iv) Circle lies completely above or below the x-axis then $g^2 < c$.

(v) Intercept cut by a line on the circle $x^2 + y^2 + 2gx + 2fy + c = 0$ or length of chord of the circle = $2\sqrt{a^2-P^2}$ where a is the radius and P is the length of perpendicular from the centre to the chord.



(d) Diametrical form of circle :

If $A(x_1, y_1)$ and $B(x_2, y_2)$ are the end points of the diameter of the circle and $P(x, y)$ is the point other then A and B on the circle then from geometry we know that $\angle APB = 90^\circ$.



$\Rightarrow$ (Slope of PA) $\times$ (Slope of PB) = -1

$\Rightarrow \left(\dfrac{y-y_1}{x-x_1}\right)\left(\dfrac{y-y_2}{x-x_2}\right) = -1 \quad \Rightarrow (x - x_1)(x - x_2) + (y - y_1)(y - y_2) = 0$

Note : This will be the circle of least radius passing through (x_1, y_1) and (x_2, y_2)

(e) The parametric forms of the circle :

(i) The parametric equation of the circle $x^2 + y^2 = r^2$ are $x = r\cos\theta$, $y = r\sin\theta$; $\theta \in [0, 2\pi]$

(ii) The parametric equation of the circle $(x - h)^2 + (y - k)^2 = r^2$ is $x = h + r\cos\theta$, $y = k + r\sin\theta$ where θ is parameter.

(iii) The parametric equation of the circle $x^2 + y^2 + 2gx + 2fy + c = 0$ are $x = -g + \sqrt{g^2+f^2-c}\,\cos\theta$

$y = -f + \sqrt{g^2+f^2-c}\,\sin\theta$ where θ is parameter.

Note that equation of a straight line joining two point α and β on the circle $x^2 + y^2 = a^2$ is

$x\cos\dfrac{\alpha+\beta}{2} + y\sin\dfrac{\alpha+\beta}{2} = a\cos\dfrac{\alpha-\beta}{2}$.

Ex.1 Find the centre and the radius of the circles **(A)** $3x^2+3y^2-8x-10y+3=0$

(B) $x^2+y^2+2x\sin\theta+2y\cos\theta-8=0$ **(C)** $2x^2+\lambda xy+2y^2+(\lambda-4)x+6y-5=0$, for some λ.

Sol. **(A)** We rewrite the given equation as $x^2+y^2-\dfrac{8}{3}x-\dfrac{10}{3}y+1=0 \Rightarrow g=-\dfrac{4}{3}, f=-\dfrac{5}{3}, c=1$

Hence the centre is $\left(\dfrac{4}{3},\dfrac{5}{3}\right)$ and the radius is $\sqrt{\dfrac{16}{9}+\dfrac{25}{9}-1}=\sqrt{\dfrac{32}{9}}=\dfrac{4\sqrt{2}}{3}$ units

(B) $x^2+y^2+2x\sin\theta+2y\cos\theta-8=0$

Centre of this circle is $(-\sin\theta, -\cos\theta)$. Radius $=\sqrt{\sin^2\theta+\cos^2\theta+8}=\sqrt{1+8}=3$ units

(C) $2x^2+\lambda xy+2y^2+(\lambda-4)x+6y-5=0$

We rewrite the equation as $x^2+\dfrac{\lambda}{2}xy+y^2+\left(\dfrac{\lambda-4}{2}\right)x+3y-\dfrac{5}{2}=0$ ...(i)

Since, there is no term of xy in the equation of circle $\Rightarrow \dfrac{\lambda}{2}=0 \Rightarrow \lambda=0$

So, equation (i) reduces to $x^2+y^2-2x+3y-\dfrac{5}{2}=0$ $\therefore$ centre is $\left(1,-\dfrac{3}{2}\right)$ & Radius $=\sqrt{1+\dfrac{9}{4}+\dfrac{5}{2}}=\dfrac{\sqrt{23}}{2}$ units

Ex.2 If the lines $3x-4y+4=0$ and $6x-8y-7=0$ are tangents to a circle, then the radius of the radius of the circle is

Sol. The diameter of the circle is perpendicular distance between the parallel lines (tangents) $3x-4y+4=0$ and $3x-4y-\dfrac{7}{2}=0$ and so it is equal to $\dfrac{4+7/2}{\sqrt{9+16}}=\dfrac{3}{2}$. Hence radius is $\dfrac{3}{4}$.

Ex.3 If y = 2x + m is a diameter to the circle $x^2+y^2+3x+4y-1=0$, then find m

Sol. Centre of circle = (−3/2, − 2). This lies on diameter y = 2x + m $\Rightarrow -2=-3/2\times2+m \Rightarrow m=1$

Ex.4 The equation of a circle which passes through the point (1, − 2) and (4, −3) and whose centre lies on the lies 3x + 4y = 7 is

Sol. Let the circle be $x^2+y^2+2gx+2fy+c=0$ ...(i)

Hence, substituting the points, (1, − 2) and (4, −3) in equation (i)

$5+2g-4f+c=0$ ...(ii)

$25+8g-6f+c=0$ ...(iii)

= centre (−g, −f) lies on line $3x+4y=7$. Hence $-3g-4f=7$

solving for g, f, c, we get $g=\frac{-47}{15}, f=\frac{9}{15}, c=\frac{55}{15}$

Hence the equation is $15(x^2+y^2)-94x+18y+55=0$

Ex.5 A circle has radius equal to 3 units and its centre lies on the line y = x − 1. Find the equation of the circle if it passes through (7, 3).

Sol. Let the centre of the circle be (α, β). It lies on the line y = x − 1

$\Rightarrow$ $\beta=\alpha-1$. Hence the centre is $(\alpha, \alpha-1)$.

$\Rightarrow$ The equation of the circle is $(x-\alpha)^2+(y-\alpha+1)^2=9$

It passes through (7, 3) $\Rightarrow (7-\alpha)^2+(4-\alpha)^2=9 \Rightarrow 2\alpha^2-22\alpha+56=0$

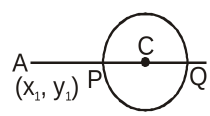
$\Rightarrow \alpha^2-11\alpha+28=0 \Rightarrow (\alpha-4)(\alpha-7)=0 \Rightarrow \alpha=4, 7$

Hence the required equations are $x^2+y^2-8x-6y+16=0$ and $x^2+y^2-14x-12y+76=0$

C. POSITION OF A POINT W.R.T. CIRCLE

(a) Let the circle is $x^2+y^2+2gx+2fy+c=0$ and the point is (x_1, y_1) then point (x_1, y_1) lies out side the circle or on the circle or inside the circle according as

$\Rightarrow x_1^2+y_1^2+2gx_1+2fy_1+c >,=,< 0$ or $S_1 >,=,< 0$



(b) The greatest & the least distance of a point A from a circle with centre C & radius r is AC + r & AC − r respectively.

Ex.6 If P(2, 8) is an interior point of a circle $x^2 + y^2 - 2x + 4y - p = 0$ which neither touches nor intersects the axes, then set for p is

Sol. For internal point P(2, 8), 4 + 64 − 4 + 32 − p < 0 $\Rightarrow$ p > 96 and x intercept $=2\sqrt{1+p}$ therefore 1+p<0

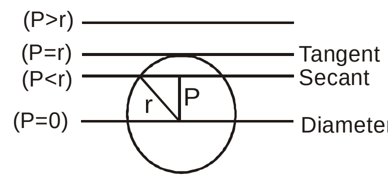
$\Rightarrow p<-1$ and y intercept $=2\sqrt{4+p} \Rightarrow p<-4$

D. TANGENT LINE OF CIRCLE

When a straight line meet a circle on two coincident points then it is called the tangent of the circle.

(a) Condition of Tangency :

The line L = 0 touches the circle S = 0 if P the length of the perpendicular from the centre to that line and radius of the circle r are equal i.e. P = r.



Ex.7 Find the range of parameter 'a' for which the variable line y = 2x + a lies between the circle $x^2 + y^2 - 2x - 2y + 1 = 0$ and $x^2 + y^2 - 16x - 2y + 61 = 0$ without intersecting or touching either circle.

Sol. The given circles are $C_1 : (x-1)^2 + (y-1)^2 = 1$ and $C_2 : (x-8)^2 + (y-1)^2 = 4$

The line $y - 2x - a = 0$ will lie between these circle if centre of the circles lie on opposite sides of the line, i.e. $(1-2-a)(1-16-a) < 0 \Rightarrow a \in (-15, -1)$

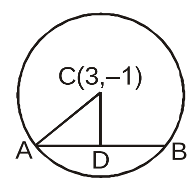
Line wouldn't touch or intersect the circles if, $\frac{|1-2-a|}{\sqrt{5}} > 1, \frac{|1-16-a|}{\sqrt{5}} > 2$

$\Rightarrow \ |1+a| > \sqrt{5}, |15+a| > 2\sqrt{5} \ \Rightarrow \ a > \sqrt{5}-1$ or $a < -\sqrt{5}-1, \ a > 2\sqrt{5}-15$ or $a < -2\sqrt{5}-15$

Hence common values of 'a' are $(2\sqrt{5}-15, -\sqrt{5}-1)$.

Ex.8 The equation of a circle whose centre is (3, –1) and which cuts off a chord of length 6 on the line 2x – 5y + 18 = 0.

Sol. Let AB(= 6) be the chord intercepted by the line 2x – 5y + 18 = 0 from the circle and let CD be the perpendicular drawn from centre (3, – 1) to the chord AB.

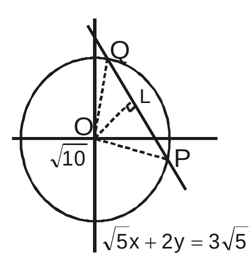


i.e., $AD = 3, CD = \frac{2.3 - 5(-1) + 18}{\sqrt{2^2 + 5^2}} = \sqrt{29}$

Therefore, $CA^2 = 3^2 + (\sqrt{29})^2 = 38$. Hence required equation is $(x-3)^2 + (y+1)^2 = 38$

Ex.9 The area of the triangle formed by line joining the origin to the points of intersection(s) of the line $x\sqrt{5} + 2y = 3\sqrt{5}$ and circle $x^2 + y^2 = 10$ is

Sol. Length of perpendicular from origin to the line $x\sqrt{5} + 2y = 3\sqrt{5}$ is



$$OL = \frac{3\sqrt{5}}{\sqrt{(\sqrt{5})^2 + 2^2}} = \frac{3\sqrt{5}}{\sqrt{9}} = \sqrt{5}$$

Radius of the given circle $= \sqrt{10} = OQ = OP$

$PQ = 2QL = 2\sqrt{OQ^2 - OL^2} = 2\sqrt{10-5} = 2\sqrt{5}$. Thus area of $\Delta OPQ = \frac{1}{2} \times PQ \times OL = \frac{1}{2} \times 2\sqrt{5} \times \sqrt{5} = 5$

(b) Equation of the tangent (T = 0) :

(i) Tangent at the point (x_1, y_1) on the circle $x^2 + y^2 = a^2$ is $xx_1 + yy_1 = a^2$.

(ii) (1) The tangent at the point (acos t, asin t) on the circle $x^2 + y^2 = a^2$ is $x\cos t + y\sin t = a$

(2) The point of intersection of the tangents at the points $P(\alpha)$ and $Q(\beta)$ is $\dfrac{a\cos\dfrac{\alpha+\beta}{2}}{\cos\dfrac{\alpha-\beta}{2}}, \dfrac{a\sin\dfrac{\alpha+\beta}{2}}{\cos\dfrac{\alpha-\beta}{2}}$

(iii) The equation of tangent at the point (x_1, y_1) on the circle $x^2 + y^2 + 2gx + 2fy + c = 0$ is

$xx_1 + yy_1 + g(x + x_1) + f(y + y_1) + c = 0$

(iv) If line y = mx + c is a straight line touching the circle $x^2 + y^2 = a^2$, then $c = \pm a\sqrt{1+m^2}$ and contact points are $\left(\mp\dfrac{am}{\sqrt{1+m^2}}, \mp\dfrac{a}{\sqrt{1+m^2}}\right)$ or $\left(\mp\dfrac{a^2m}{c}, \pm\dfrac{a^2}{c}\right)$ and equation of tangent is $y = mx \pm a\sqrt{1+m^2}$

(v) The equation of tangent with slope m of the circle $(x-h)^2 + (y-k)^2 = a^2$ is

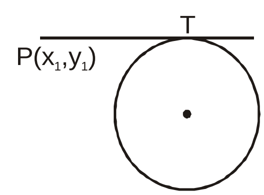
$(u-k) = m(x-h) \pm a\sqrt{1+m^2}$

Note : To get the equation of tangent at the point (x_1, y_1) on any curve we replace xx_1 in place of x^2, yy_1 in place of y^2, $\dfrac{x+x_1}{2}$ in place of x, $\dfrac{y+y_1}{2}$ in place of y, $\dfrac{xy_1+yx_1}{2}$ in place of xy and c in place of c.

(c) Length of tangent ($\sqrt{S_1}$) :

The length of tangent draw from point (x_1, y_1) out side the circle $S \equiv x^2 + y^2 + 2gx + 2fy + c = 0$ is,

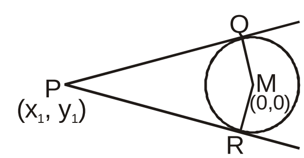
$PT = \sqrt{S_1} = \sqrt{x_1^2 + y_1^2 + 2gx_1 + 2fy_1 + c}$



Note : When we use this formula the coefficient of x^2 and y^2 must be 1.

(d) Equation of Pair to tangents ($SS_1 = T^2$) :

Let the equation of circle $X \equiv x^2 + y^2 = a^2$ and $P(x_1, y_1)$ is any point outside the circle. From the point we can draw two real and distinct tangent PQ & PR and combine equation of pair of tangents is $(x^2 + y^2 - a^2)(x_1^2 + y_1^2 - a^2) = (xx_1 + yy_1 - a^2)^2$ or $SS_1 = T^2$

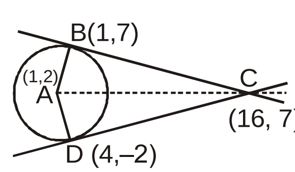


Ex.10 Let A be the centre of the circle $x^2 + y^2 - 2x - 4y - 20 = 0$ and B(1, 7) and D(4, −2) are points on the circle then, if tangents be drawn at B and D, which meet at C, then area of quadrilateral ABCD is

Sol. $x.1 + y.7 - 1(x+1) - 2(y+7) - 20 = 0$ or y = 7 ...(i)

Tangent at $D(4, -2)$ is $3x - 4y - 20 = 0$ ...(ii)

Solving (i) and (ii), C is (16, 7)
Area ABCD = AB × BC = 5 × 15 = 75 units.

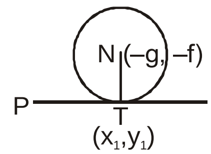


E. NORMAL OF CIRCLE

Normal at a point of the circle is the straight line which is perpendicular to the tangent at the point of contact and passes through the centre of circle.

(a) Equation of normal at point (x_1, y_1)

or circle $x^2 + y^2 + 2gx + 2fy + c = 0$ is



$$y - y_1 = \left(\frac{y_1 + f}{x_1 + g}\right)(x - x_1)$$

(b) The equation of normal on any point (x_1, y_1) of circle $x^2 + y^2 = a^2$ is $\left(\frac{y}{x} = \frac{y_1}{x_1}\right)$

(c) If $x^2 + y^2 = a^2$ is the equation of the circle then at any point 't' of this circle (a cos t, a sin t), the equation of normal is x sin t – y cos t = 0.

Ex.11 Find the equation of the normal to the circle $x^2 + y^2 - 5x + 2y - 48 = 0$ at the point (5, 6).

Sol. Since normal to the circle always passes through the centre so equation of the normal will be the line passing through (5, 6) & $\left(\frac{5}{2}, -1\right)$ i.e. $y + 1 = \frac{7}{5/2}\left(x - \frac{5}{2}\right) \Rightarrow 5y + 5 = 14x - 35 \Rightarrow 14x - 5y - 40 = 0$

Ex.12 If the straight line ax + by = 2; a, b ≠ 0 touches the circle $x^2 + y^2 - 2x = 3$ and is normal to the circle $x^2 + y^2 - 4y = 6$, then the values of a and b are respectively

Sol. Given $x^2 + y^2 - 2x = 3$ $\quad\therefore$ centre is (1, 0) and radius is 2 and $x^2 + y^2 - 4y = 6$

Given $x^2 + y^2 - 4y = 6$

$\therefore$ centre is (0, 2) and radius is $\sqrt{10}$. Since line ax + by = 2 touches the first circle

$\therefore \dfrac{a(1) + b(0) - 2}{\sqrt{a^2 + b^2}} = 2$ or $(a - 2) = \left[2\sqrt{a^2 + b^2}\right]$ ....(i)

Also the given line is normal to the second circle. Hence it will pass through the centre of the second circle.

$\therefore$ a(0) + b(B) = 2 or 2b = 2 or b = 1

Putting this value in equation (i) we get $a - 2 = 2\sqrt{a^2 + 1^2}$ or $(a - 2)^2 = 4(a^2 + 1)$

or $a^2 + 4 - 4a = 4a^2 + 4$ or $3a^2 + 4a = 0$ or $a(3a + 4) = 0$ or $a = 0, -\frac{4}{3}$

$\therefore$ values of a and b are $\left(-\frac{4}{3}, 1\right)$ respectively according to the given choices.

Ex.13 Find the equation of a circle having the lines $x^2 + 2xy + 3x + 6y = 0$ as its normal and having size just sufficient to contain the circle x(x – 4) + y(y – 3) = 0.

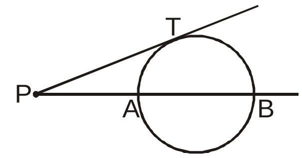
Sol. Pair of normals are (x + 2y)(x + 3) = 0 $\therefore$ Normals are x + 2y = 0, x + 3 = 0.

Point of intersection of normals is the centre of required circle i.e. $C_1(-3, 3/2)$ and centre of given circle is $C_2(2, 3/2)$ and radius $r_2 = \sqrt{4 + \frac{9}{4}} = \frac{5}{2}$

Let r_1 be the radius of required circle $\Rightarrow r_1 = C_1C_2 + r_2 = \sqrt{(-3-2)^2 + \left(\frac{3}{2} - \frac{3}{2}\right)^2} + \frac{5}{2} = \frac{15}{2}$

Hence equation of required circle is $x^2 + y^2 + 6x - 3y - 45 = 0$

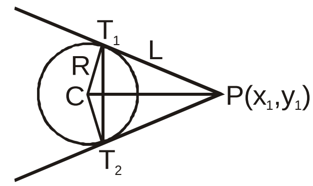
F. POWER OF THE POINT



square of the length of the tangent from the point P is defined as power of the point 'p' w.r. to given circle $\Rightarrow PT^2 = S_1$

Note :- Power of a point remains constant w.r. to a circle $PA \cdot PB = PT^2$

G. CHORD OF CONTACT



A line joining the two points of contacts of two tangents drawn from a point out side the circle, is called chord of contact of that point.

If two tangents PT_1 & PT_2 are drawn from the point $P(x_1, y_1)$ to the circle $S \equiv x^2 + y^2 + 2gx + 2fy + c = 0$, then the equation of the chord of contact T_1T_2 is : $xx_1 + yy_1 + g(x + x_1) + f(y + y_1) + c = 0$

(i.e. T = 0 same as equation of tangent).

Remember :

(a) Length of chord of contact $T_1 T_2 = \dfrac{2LR}{\sqrt{R^2 + L^2}}$.

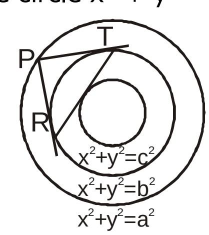
(b) Area of the triangle formed by the pair of the tangents & its chord of contact $= \dfrac{RL^3}{R^2 + L^2}$ Where R is the radius of the circle & L is the length of the tangent from (x_1, y_1) on $S = 0$.

(c) Angle between the pair of tangents from $P(x_1, y_1) = \tan^{-1}\left(\dfrac{2RL}{L^2 - R^2}\right)$

(d) Equation of the circle circumscribing the triangle PT_1T_2 or quadrilateral CT_1PT_2 is : $(x - x_1)(x + g) + (y - y_1)(y + f) = 0$.

(e) The joint equation of a pair of tangents drawn from the point A (x_1, y_1) to the circle $x^2 + y^2 + 2gx + 2fy + c = 0$ is : $SS_1 = T^2$.
Where $S \equiv x^2 + y^2 + 2gx + 2fy + c$; $S_1 \equiv x_1^2 + y_1^2 + 2gx_1 + 2fy_1 + c$
$T \equiv xx_1 + yy_1 + g(x + x_1) + f(y + y_1) + c$.

Ex.14 The chord of contact of tangents drawn from a point on the circle $x^2 + y^2 = a^2$ to the circle $x^2 + y^2 = b^2$ touches the circle $x^2 + y^2 = c^2$. Show that a, b, c are in G.P.



Sol. Let $P(a\cos\theta, a\sin\theta)$ be a point on the circle $x^2 + y^2 = a^2$.
The equation of chord of contact of tangents drawn from $P(a\cos\theta, a\sin\theta)$ to the circle $x^2 + y^2 = b^2$ is $ax\cos\theta + ay\sin\theta = b^2$ .....(i)
This touches the circle $x^2 + y^2 = c^2$ .....(ii)
$\therefore$ Length of perpendicular from (0, 0) to (i) = radius of (ii)

$$\therefore \frac{|0 + 0 - b^2|}{\sqrt{(a^2\cos^2\theta + a^2\sin^2\theta)}} = c \quad \text{or} \quad b^2 = ac \qquad \Rightarrow \text{a, b, c are in G.P.}$$

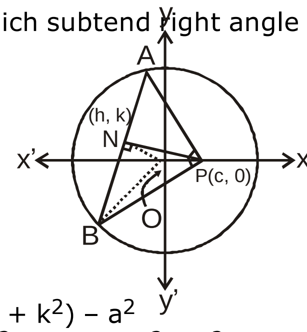
H. EQUATION OF THE CHORD WITH A GIVEN MIDDLE POINT ($T = S_1$)

The equation of the chord of the circle $S \equiv x^2 + y^2 + 2gx + 2fy + c = 0$ in terms of its mid point $M(x_1, y_1)$ is $y - y_1 = -\dfrac{x_1 + g}{y_1 + f}(x - x_1)$. This on simplification can be put in the form

$xx_1 + yy_1 + g(x + x_1) + f(y + y_1) + c = x_1^2 + y_1^2 + 2gx_1 + 2fy_1 + c$ which is designated by $T = S_1$.

Note that : The shortest chord of a circle passing through a point 'M' inside the circle, is one chord whose middle point is M.

Ex.15 Find the locus of middle points of chords of the circle $x^2 + y^2 = a^2$, which subtend right angle at the point (c, 0).



Sol. Let N(h, k) be the middle point of any chord AB, which subtend a right angle at P(c, 0).
Since $\angle APB = 90^o$ $\therefore$ NA = NB = NP
(since distance of the vertices from middle point of the hypotenuse are equal)
or $(NA)^2 = (NB)^2 = (h - c)^2 + (k - 0)^2$ ....(i)
But also $\angle BNO = 90^o$ $\therefore$ $(OB)^2 = (ON)^2 + (NB)^2$
$\Rightarrow$ $-(NB)^2 = (ON)^2 - (NB)^2$ $\Rightarrow$ $-[(h - c)^2 + (k - 0)^2] = (h^2 + k^2) - a^2$
or $2(h^2 + k^2) - 2ch + c^2 - a^2 = 0$ $\therefore$ Locus of N(h, k) is $2(x^2 + y^2) - 2cx + c^2 - a^2 = 0$

Ex.16 Let a circle be given by $2x(x - a) + y(2y - b) = 0$ $(a \neq 0, b \neq 0)$ Find the condition on a and b if two chords, each bisected by the x-axis, can be drawn to the circle from (a, b/2).

Sol. The given circle is $2x(x - a) + y(2y - b) = 0$ or $x^2 + y^2 - ax - by/2 = 0$
Let AB be the chord which is bisected by x-axis at a point M. Let its co-ordinates be M(h, 0).
and $S \equiv x^2 + y^2 - ax - by/2 = 0$ $\therefore$ Equation of chord AB is $T = S_1$

$$hx + 0 - \frac{a}{2}(x + h) - \frac{b}{4}(y + 0) = h^2 + 0 - ah - 0$$

Since its passes through (a, b/2) we have $ah - \frac{a}{2}(a + h) - \frac{b^2}{8} = h^2 - ah \Rightarrow h^2 - \frac{3ah}{2} + \frac{a^2}{2} + \frac{b^2}{8} = 0$

Now there are two chords bisected by the x-axis, so there must be two distinct real roots of h.

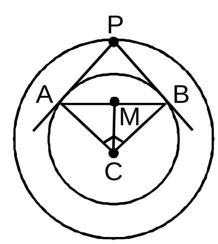
$$\therefore \quad B^2 - 4AC > 0 \quad \Rightarrow \quad \left(\frac{-3a}{2}\right)^2 - 4.1.\left(\frac{a^2}{2} + \frac{b^2}{8}\right) > 0 \quad \Rightarrow \quad a^2 > 2b^2.$$

I. DIRECTOR CIRCLE

The locus of point of intersection of two perpendicular tangents to a circle is called director circle. Let P(h, k) is the point of intersection of two tangents drawn on the circle $x^2 + y^2 = a^2$. Then the equation of the pair of tangents is $SS_1 = T^2$. i.e. $(x^2 + y^2 - a^2)(h^2 + k^2 - a^2) = (hx + ky - a^2)^2$
As lines are perpendicular to each then, coefficient of x^2 + coefficient of $y^2 = 0$.
$\Rightarrow$ $[(h^2 + k^2 - a^2) - h_2] + [(h^2 + k^2 - a^2) - k^2] = 0$ $\Rightarrow$ $h^2 + k^2 = 2a^2$
$\therefore$ locus of (h, k) is $x^2 + y^2 = 2a^2$ which is the equation of the director circle.
$\therefore$ director circle is a concentric circle whose radius is $\sqrt{2}$ times the radius of the circle.
Note : The director circle of $x^2 + y^2 + 2gx + 2fy + c = 0$ is $x^2 + y^2 + 2gx + 2fy + 2c - g^2 - f^2 = 0$

Ex.17 Let P be any moving point on the circle $x^2 + y^2 - 2x = 1$, from this point chord of contact is drawn w.r.t the circle $x^2 + y^2 - 2x = 0$. Find the locus of the circumcentre of the triangle CAB, C being centre of the circle.

Sol. : The two circles are $(x - 1)^2 + y^2 = 1$ ....(i) $(x - 1)^2 + y^2 = 2$ ....(ii)
So the second circle is the director circle of the first. So $\angle APB = \pi/2$. Also $\angle ACB = \pi/2$
Now circumcentre of the right angled triangle CAB would lie on the mid point of AB
So let the point be $M \equiv (h, k)$

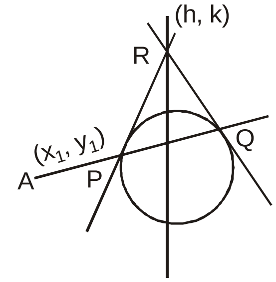


Now, $CM = CB \sin 45^o = \frac{1}{\sqrt{2}}$

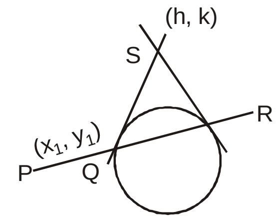
So, $(h - 1)^2 + k^2 = \left(\frac{1}{\sqrt{2}}\right)$

So, locus of M is $(x - 1)^2 + y^2 = \frac{1}{2}$.

J. POLE AND POLAR



Let any straight line through the given point $A(x_1, y_1)$ intersect the given circle S = 0 in two points P' and Q and if the tangent of the circle at P and Q meet at the point R then locus of point R is called polar of the point A and point A is called the pole, with respect to the given circle.



(a) The equation of the polar of point (x_1, y_1) w.r.t. circle $x^2 + y^2 = a^2$.

Let PQR is a chord which passes through the point $P(x_1, y_1)$ which intersects the circle at point Q and R and the tangents are drawn at points Q and R meet at point S(h, k) then equation of QR the chord of contact is $x_1h + y_1k = a^2$

$\therefore$ locus of point S(h, k) is $xx_1 + yy_1 = a^2$ which is the equation of the polar.

Notes :

(i) The equation of the polar is the T = 0, so the polar of point (x_1, y_1) w.r.t. circle $x^2 + y^2 + 2gx + 2fy + c = 0$ is $xx_1 + yy_1 + g(x + x_1) + f(y + y_1) + c = 0$.

(ii) If point is outside the circle then equation of polar and chord of contact is same. So the chord of contact is polar.

(iii) If point is inside the circle then chord of contact does not exist but polar exists.

(iv) If point lies on the circle then polar, chord of contact and tangent on that point are same.

(v) If the polar of P w.r.t. a circle passes through the point Q, then the polar of point Q will pass through P and hence P and Q are conjugate points of each other w.r.t. the given circle.

(vi) If pole of a line w.r.t. a circle lies on second line. Then pole of second line lies on first line and hence both lines are conjugate lines of each other w.r.t. the given circle.

(b) Pole of a given line with respect to a circle.

To find the pole of a line we assume the coordinates of the pole then from these coordinates we find the polar. This polar and given line represent the same line. Then by comparing the coefficients of similar terms we can get the coordinates of the pole. The pole of $\ell x + my + n = 0$. w.r.t. circle $x^2 + y^2 = a^2$ will be $\left(\frac{-\ell a^2}{n}, \frac{-ma^2}{n}\right)$.

K. FAMILY OF CIRCLES

(a) The equation of the family of circles passing through the points of intersection of two circles $S_1 = 0$ and $S_2 = 0$ is ; $S_1 + KS_2 = 0$ $(K \neq -1)$.

(b) The equation of the family of circles passing through the point of intersection of a circle S = 0 and line L = 0 is given by S + KL = 0,

(c) The equation of a family of circles passing through two given points (x_1, y_1) and (x_2, y_2) can be written in the form $(x - x_1)(x - x_2) + (y - y_1)(y - y_2) + K\begin{vmatrix} x & y & 1 \\ x_1 & y_1 & 1 \\ x_2 & y_2 & 1 \end{vmatrix} = 0$ where K is a parameter.

(d) The equation of a family of circles touching a fixed line $y - y_1 = m(x - x_1)$ at the fixed point (x_1, y_1) is $(x - x_1)^2 + (y - y_1)^2 + K[y - y_1 - m(x - x_1)] = 0$, where K is a parameter.

(e) Family of circles cicumscribing a triangle whose sides are given by $L_1 = 0$; $L_2 = 0$ and $L_3 = 0$ is given by ; $L_1L_2 + \lambda L_2L_3 + \mu L_3L_1 = 0$ provided coefficient of $xy = 0$ and coefficient of x^2 = coefficient of y^2

(f) Equation of circle circumscribing a quadrilateral whose side in order are represented by the line $L_1=0$, $L_2 = 0$, $L_3 = 0$ and $L_4 = 0$ are $L_1L_3 + \lambda L_2L_4 = 0$ provided coefficient of x^2 = coefficient of y^2 and coefficient of $xy = 0$.

Ex.18 The equation of the circle through the points of intersection of $x^2 + y^2 - 1 = 0$, $x^2 + y^2 - 2x - 4y + 1 = 0$ and touching the line $x + 2y = 0$, is

Sol. Family of circle is $x^2 + y^2 - 2x - 4y + 1 + \lambda(x^2 + y^2 - 1) = 0$

$$(1+\lambda)x^2 + (1+\lambda)y^2 - 2x - 4y + (1-\lambda) = 0 \quad \text{or} \quad x^2 + y^2 - \frac{2}{1+\lambda}x - \frac{4}{1+\lambda}y + \frac{1-\lambda}{1+\lambda} = 0$$

Centre is $\left[\frac{1}{1+\lambda}, \frac{2}{1+\lambda}\right]$ and radius = $\sqrt{\left(\frac{1}{1+\lambda}\right)^2 + \left(\frac{2}{1+\lambda}\right)^2 - \frac{1-\lambda}{1+\lambda}} = \frac{\sqrt{4+\lambda^2}}{1+\lambda}$

Since it touches the line $x + 2y = 0$, hence, Radius = Perpendicular distance from centre to the line.

i.e., $$\left|\frac{\frac{1}{1+\lambda} + 2\frac{2}{1+\lambda}}{\sqrt{1^2 + 2^2}}\right| = \frac{\sqrt{4+\lambda^2}}{1+\lambda} \Rightarrow \sqrt{5} = \sqrt{4+\lambda^2} \Rightarrow \lambda = \pm 1$$

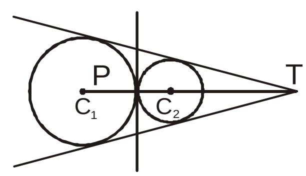
$\lambda = -1$ cannot be possible in case of circle. So $\lambda = 1$. Thus, we get the equation of circle

L. DIRECT AND TRANSVERSE COMMON TANGENTS

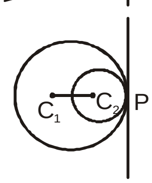
Let two circles having centre C_1 and C_2 and radii, r_1 and r_2 and C_1C_2 is the distance between their centres then

(a) Both circles will touch :

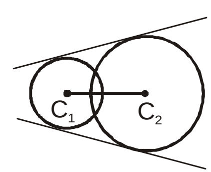
(i) Externally : If $c_1c_2 = r_1 + r_2$ i.e. the distance between their centres is equal to sum of their radii and point P divides C_1C_2 in the ratio $r_1 : r_2$ (internally). In this case there will be **three common tangents.**



(ii) Internally : If $C_1C_2 = |r_1 - r_2|$ i.e. the distance between their centres is equal to difference between their radii and point P divides C_1C_2 in the ratio $r_1 : r_2$ **externally** and in this case there will be **only one common tangent.**

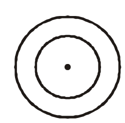


(b) The circles will intersect : when $|r_1 - r_2| < C_1C_2 < r_1 + r_2$ in this case there will be **two common tangents.**

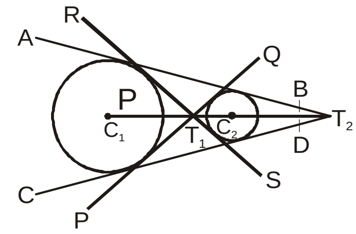


(c) The circles will not intersect :

(i) One circle will lie inside the other circle if $C_1C_2 < |r_1 - r_2|$
In this case there will be **no common tangent.**



(ii) When circle are apart from each other the $C_1C_2 > r_1 + r_2$ and in this case there will be four common tangents. Lines PQ and RS are called **transverse or indirect or internal common tangents** and these lines meet line C_1C_2 on T_1 and T_1 divides the line C_1C_2 in the ratio $r_1 : r_2$ internally and lines AB & CD are called **direct or external common tangents.** These lines meet C_1C_2 produced on T_2. Thus T_2 divides C_1C_2 externally in the ratio $r_1 : r_2$.



Note : Length of direct common tangent = $\sqrt{(C_1C_2)^2 - (r_1 - r_2)^2}$

Length of transverse common tangent = $\sqrt{(C_1C_2)^2 - (r_1 + r_2)^2}$

Ex.19 Prove that the circles $x^2 + y^2 + 2ax + c^2 = 0$ and $x^2 + y^2 + 2by + c^2 = 0$ touch each other, if $\frac{1}{a^2} + \frac{1}{b^2} = \frac{1}{c^2}$.

Sol. Given circles are $x^2 + y^2 + 2ax + c^2 = 0$ ...(i) and $x^2 + y^2 + 2by + c^2 = 0$ ...(ii)

Let C_1 and C_2 be the centre of circle (i) and (ii), respectively and r_1 and r_2 be their radii, then

$C_1 = (-a, 0),\ C_2 = (0, -b)$, $r_1 = \sqrt{a^2 - c^2}$, $r_2 = \sqrt{b^2 - c^2}$

Here we find the two circles touch each other internally or externally.

For touch, $|C_1C_2| = |r_1 \pm r_2|$ or $\sqrt{(a^2 + b^2)} = \left|\sqrt{(a^2 - c^2)} \pm \sqrt{(b^2 - c^2)}\right|$

On squaring $a^2 + b^2 = a^2 - c^2 + b^2 - c^2 \pm 2\sqrt{(a^2 - c^2)}\sqrt{(b^2 - c^2)}$ or $c^2 = \pm\sqrt{a^2b^2 - c^2(a^2 + b^2) + c^4}$

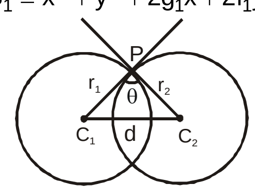
Again squaring, $c^4 = a^2b^2 - c^2(a^2 + b^2) + c^4$ or $c^2(a^2 + b^2) = a^2b^2$ or $\frac{1}{a^2} + \frac{1}{b^2} = \frac{1}{c^2}$

M. THE ANGLE OF INTERSECTION OF TWO CIRCLES

Definition : The angle between the tangents of two circles at the point of intersection of the two circles is called angle of intersection of two circles. If two circles are $S_1 \equiv x^2 + y^2 + 2g_1x + 2f_1y + c_1 = 0$

$S_2 \equiv x^2 + y^2 + 2g_2x + 2f_2y + c_2 = 0$ and θ is the angle between them

then $\cos\theta = \frac{2g_1g_2 + 2f_1f_2 - c_1 - c_2}{\sqrt{g_1^2 + f_1^2 - c_1}\ \sqrt{g_2^2 + f_2^2 - c_2}}$ or $\cos\theta = \left(\frac{r_1^2 + r_2^2 - d^2}{2r_1r_2}\right)$



Here a_1 and a_2 are the radii of the circles and d is the distance between their centres.

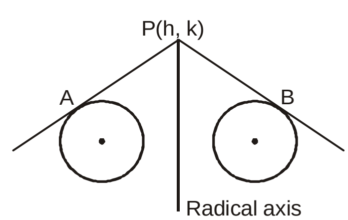
If the angle of intersection of the two circles is a right angle then such circles are called **"Orthogonal circles"** and conditions for the circles to be orthogonal is $2g_1g_2 + 2f_1f_2 = c_1 + c_2$

N. RADICAL AXIS OF THE TWO CIRCLES ($S_1 - S_2 = 0$)

(a) Definition : The locus of a point, which moves in such a way that the length of tangents drawn from it to the circles are equal and is called the radical axis. It two circles are -

$S_1 \equiv x^2 + y^2 + 2g_1x + 2f_1y + c_1 = 0$

$S_2 \equiv x^2 + y^2 + 2g_2x + 2f_2y + c_2 = 0$



Let P(h, k) is a point and PA, PB are length of two tangents on the circles from point P. Then from definition : $\sqrt{h^2+k^2+2g_1h+2f_1k+c_1} = \sqrt{h^2+k^2+2g_2h+2f_2k+c_2}$ or $2(g_1-g_2)h+2(f_1-f_2)k+c_1-c_2=0$

$\therefore$ locus of (h, k) $\Rightarrow$ $2x(g_1-g_2)+2y(f_1-f_2)k+c_1-c_2=0$ $\Rightarrow$ $\mathbf{S_1 - S_2 = 0}$

which is the equation of radical axis.

Notes :

(i) To get the equation of the radical axis first of all make the coefficient of x^2 and y^2 = 1

(ii) If circles touch each other then Radical axis is the common tangent to both the circles.

(iii) When the two circles intersect on real points then common chord is the Radical axis of the two circles.

(iv) The Radical axis of the two circles is perpendicular to the line joining the centre of two circle of two circles but not always pass through mid point of it.

(v) The Radical axis of three circles (Taking two at a time) meet on a point.

(vi) If circles are concentric then the Radical axis does not always exist but if circles are not concentric then Radical axis always exists.

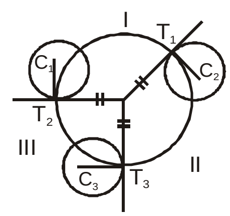
(vii) If two circles are orthogonal to the third circle then Radical axis of both circles passes through the centre of the third circle.

(viii) A system of circle, every pair of which have the same radical axis, is called a coaxial system of circles.

(b) Radical centre

The radical centre of three circles is the point form which length of tangents on three circles are equal i.e. the point of intersection of radical axis of the circles is the radical centre of the circles. To get the radical axis of three circles $S_1 = 0$, $S_2 = 0$, $S_3 = 0$ we have to solve any two $S_1 - S_2 = 0$, $S_2 - S_3 = 0$, $S_3 - S_1 = 0$

Notes :



(i) The circle with centre as radical centre and radius equal to the length of tangent from radical centre to any of the circle, will cut the three circles orthogonally.

(ii) If three circles are drawn on three sides of a triangle taking them as diameter then its orthocentre will be its radical centre.

(iii) Locus of the centre of a variable circle orthogonal to two fixed circles is the radical axis between the two fixed circles.

(iv) If two circles are orthogonal, then the polar of a point 'P' on first circle w.r.t. the second circle passes through the point Q which is the other end of the diameter through P. Hence locus of a point which moves such that its polars w.r.t. the circles, $S_1 = 0$, $S_2 = 0$ & $S_3 = 0$ are concurrent is a circle which is orthogonal to all the three circles.

Ex.20 A and B are two fixed points and P moves such that PA = nPB where $n \neq 1$. Show that locus of P is a circle and for different values of n all the circles have a common radical axis.

Sol. Let $A \equiv (a,\ 0)$, $B \equiv (-a,\ 0)$ and P(h, k) so PA = nPB $\Rightarrow$ $(h-a)^2+k^2 = n^2[(h+a)^2+k^2]$

$\Rightarrow$ $(1-n^2)h^2+(1-n^2)k^2-2ah(1+n^2)+(1-n^2)a^2=0$ $\Rightarrow$ $h^2+k^2-2ah\left(\frac{1+n^2}{1-n^2}\right)+a^2=0$

Hence locus of P is $x^2+y^2-2ax\left(\frac{1+n^2}{1-n^2}\right)+a^2=0$, which is a circle of different values of n.

Let n_1 and n_2 are two different values of n so their radical axis is x = 0 i.e. y-axis. Hence for different values of n the circle have a common radical axis.

Ex.21 Find the equation of the circle through the points of intersection of the circles $x^2 + y^2 - 4x - 6y - 12 = 0$ and $x^2 + y^2 + 6x + 4y - 12 = 0$ and cutting the circle $x^2 + y^2 - 2x - 4 = 0$ orthogonally.

Sol. The equation of the circle through the intersection of the given circles is

$$x^2 + y^2 - 4x - 6y - 12 + \lambda(-10x - 10y) = 0 \qquad \text{...(i)}$$

where $(-10x - 10y = 0)$ is the equation of radical axis for the circle

$x^2 + y^2 - 4x - 6y - 12 = 0$ and $x^2 + y^2 + 6x + 4y - 12 = 0$

Equation (i) can be rearranged as $x^2 + y^2 - x(10\lambda + 4) - y(10\lambda + 6) - 12 = 0$

It cuts the circle $x^2 + y^2 - 2x - 4 = 0$ orthogonally.

Hence $2gg_1 + 2ff_1 = c + c_1 \Rightarrow 2(5\lambda + 2)(1) + 2(5\lambda + 3)(0) = -12 - 4 \Rightarrow \lambda = -2$

Hence the required circle is $x^2 + y^2 - 4x - 6y - 12 - 2(-10x - 10y) = 0$ i.e., $x^2 + y^2 + 16x + 14y - 12 = 0$

Ex.22 Find the radical centre of circle $x^2 + y^2 + 3x + 2y + 1 = 0$, $x^2 + y^2 - x + 6y + 5 = 0$ and $x^2 + y^2 + 5x - 8y + 15 = 0$. Also find the equation of the circle cutting them orthogonally.

Sol. Given circles are

$$S_1 \equiv x^2 + y^2 + 3x + 2y + 1 = 0$$
$$S_2 \equiv x^2 + y^2 - x + 6y + 5 = 0$$
$$S_3 \equiv x^2 + y^2 + 5x - 8y + 15 = 0$$

Equation of two radical axes are $S_1 - S_2 \equiv 4x - 4y - 4 = 0$ or $x - y - 1 = 0$

and $S_2 - S_3 \equiv -6x + 14y - 10 = 0$ or $3x - 7y + 5 = 0$

Solving them the radical centre is (3, 2) also, if r is the length of the tangent drawn from the radical centre (3, 2) to any one of the given circles, S_1, we have $r = \sqrt{S_1} = \sqrt{3^2 + 2^2 + 3.3 + 2.2 + 1} = \sqrt{27}$

Hence (3, 2) is the centre and $\sqrt{27}$ is the radius of the circle intersecting them orthogonally.

$\therefore$ Its equation is $(x - 3)^2 + (y - 2)^2 = r^2 = 27 \Rightarrow x^2 + y^2 - 6x - 4y - 14 = 0$

Alternative Method :

Let $x^2 + y^2 + 2gx + 2fy + c = 0$ be the equation of the circle cutting the given circles orthogonally.

$\therefore \quad 2g\left(\frac{3}{2}\right) + 2f(1) = c + 1$ or $3g + 2f = c + 1$ ...(i)

$2g\left(-\frac{1}{2}\right) + 2f(3) = c + 5$ or $-g + 6f = c + 5$ ...(ii)

and $2g\left(\frac{5}{2}\right) + 2f(-4) = c + 15$ or $5g - 8f = c + 15$ ...(iii)

Solving (i), (ii) and (iii) we get $g = -3, f = -2$ and $c = -14$

$\therefore$ equation of required circle is $x^2 + y^2 - 6x - 4y - 14 = 0$